

SECONDARY PERTURBATION EFFECTS IN KEPLERIAN ACCRETION DISKS: ELLIPTICAL INSTABILITY

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Origin of turbulence in cold accretion disks, particularly in 3D, which is expected to be hydrodynamic but not magnetohydrodynamic, is a big puzzle. While the flow must exhibit some turbulence in support of the transfer of mass inward and angular momentum outward, according to the linear perturbation theory it should always be stable. We demonstrate that the 3D secondary disturbance to the primarily perturbed disk which exhibits elliptical vortices into the system solves the problem. This result is essentially applicable to the outer region of accretion disks in active galactic nuclei where the gas is significantly cold and neutral in charge and the magnetic Reynolds number is smaller than 10^4 .

1. Introduction

Despite much effort devoted, the origin of hydrodynamic turbulence in Keplerian accretion disks is still poorly understood. This is essentially important for accretion disks around quiescent cataclysmic variables, proto-planetary and star-forming disks, and the outer regions of disks in active galactic nuclei.¹

A Keplerian accretion disk flow having a very low molecular viscosity must generate turbulence and successively diffusive viscosity to support transfer of mass inward. However, theoretically this flow never exhibits any unstable mode. On the other hand, laboratory experiments of Taylor-Couette systems, which are similar to Keplerian disks, seem to indicate that although the Coriolis force delays the onset of turbulence, the flow is ultimately unstable to turbulence.² However, some other experiments say against it.³ We believe that not finding hydrodynamic turbulence is due to their choice of large aspect ratio and small Reynolds number.

Various kinds of secondary instability, such as the elliptical instability, are widely discussed as a possible route to self-sustained turbulence in linearly perturbed shear flows (see, e.g.⁴). These effects have been proposed as a generic mechanism for the breakdown of many two-dimensional high Reynolds number flows. However, such effects have not been discussed properly in literatures for rotating Keplerian flows.

Therefore, we plan to show that the three-dimensional secondary perturbation can generate large growth in the flow time scale and presumably trigger turbulence in the Keplerian disk. Unlike the two-dimensional transient growth studied earlier⁵ which were shown to be killed immediately in presence of vertical structure, in the present case we demonstrate essentially three-dimensional growth. Possibility of large growth in shear flows *with rotation* by a three-dimensional perturbation opens a new window to explain hydrodynamic turbulence which has an important implication not only in astrophysics but also in general physics and fluid dynamics.

2. Perturbation equations

We consider a small section of the Keplerian disk in the shearing box approximation⁵ with the unperturbed velocity $\vec{U}_p = (0, -x, 0)$. We also assume that the incompressible flow extends from $x = -1$ to $+1$ with no-slip boundary conditions and is unbounded with periodic boundary condition along y and z . Therefore the linearized Navier-Stokes equations for a 2D *primary perturbation* such that $U_x^p \rightarrow w_x(x, y, z, t)$, $U_y^p \rightarrow U^p(x) + w_y(x, y, z, t)$, and pressure $\bar{p} \rightarrow \bar{p} + p(x, y, z, t)$ are given by

$$\frac{dw_x}{dt} = 2\Omega w_y - \frac{\partial p}{\partial x} + \frac{1}{R_e} \nabla^2 w_x, \quad \frac{dw_y}{dt} = \Omega(q-2)w_x - \frac{\partial p}{\partial y} + \frac{1}{R_e} \nabla^2 w_y, \quad (1)$$

with the equation of continuity $\partial w_x / \partial x + \partial w_y / \partial y = 0$, where the angular frequency $\Omega = 1/q$ and $q = 3/2$ for a Keplerian disk. When the Reynolds number R_e is very large, the solution of above equations is

$$w_x = \zeta \frac{k_y}{l^2} \sin(k_x x + k_y y), \quad w_y = -\zeta \frac{k_x}{l^2} \sin(k_x x + k_y y), \quad l^2 = k_x^2 + k_y^2, \quad (2)$$

where ζ is the amplitude of vorticity perturbation. Under this *primary perturbation*, the flow velocity and pressure modify to

$$\vec{U} = \vec{U}^p + \vec{w} = (w_x, -x + w_y, 0) = \mathbf{A} \cdot \vec{d}, \quad p' = \bar{p} + p(x, y, z, t), \quad (3)$$

where \vec{d} is the position vector and \mathbf{A} is a tensor of rank 2.

Now we concentrate on a further small patch of the primarily perturbed flow such that $x \ll 1/k_x$, $y \ll 1/k_y$. Then we consider a *secondary perturbation* to this flow such that $\vec{U} \rightarrow \vec{U} + \vec{u}$ and $p' \rightarrow p' + \tilde{p}$ with

$$(u_i, \tilde{p}) = (v_i(t), p(t)) \exp(ik_m(t)x^m), \quad i, m = 1, 2, 3. \quad (4)$$

Therefore, we obtain the evolution of a linearized *secondary perturbation*

$$\dot{v}_j + A_j^k v_k + 2\epsilon_{mkj} \Omega^m v^k = -ip k_j - \frac{v_j}{R_e} k^2, \quad (5)$$

$$\dot{k}_j = -(A_j^m)^T k_m, \quad k^n \dot{v}_n = k^m A_m^n v_n, \quad k^2 = k_m k^m. \quad (6)$$

3. Solution

We specifically concentrate on the flow having low viscosity. Therefore, the general solution⁶ of eqn. (5) can be written as a linear superposition of the Floquet modes

$$v_i(t) = \exp(\sigma t) f_i(\phi), \quad (7)$$

where $\phi = \varpi t$, $f_i(\phi)$ is a periodic function having time-period $T = 2\pi/\varpi$, and σ is the Floquet exponent which is different at different $\epsilon = (k_x/l)^2$. Clearly, if σ is positive then the system is unstable and plausibly turbulent.

When the secondary perturbation evolves much rapidly than the primary one and $k_i(0) = (0, 0, 1)$, σ has an analytic solution given by

$$\sigma = \sqrt{\zeta\epsilon - (2\Omega_3 - 1)(2\Omega_3 - \zeta)}. \quad (8)$$

Clearly, for a Keplerian disk $\sigma = \sqrt{\zeta\epsilon - (4 - 3\zeta)/9}$. Therefore, a Keplerian flow is hydrodynamically unstable under a vertical secondary perturbation if $\zeta > 1/3$. For other perturbations σ can be computed numerically described in detail elsewhere.⁶

We now plan to quantify this by computing the corresponding turbulent viscosity. For the isotropic disk fluid, the turbulent viscosity $\nu_t = \alpha c_s h$,⁷ where c_s and h are local sound speed and disk thickness respectively. On the other hand, shearing stress $T_{xy} = \langle u_1 u_2 \rangle = -\nu_t q \Omega$. Therefore, we obtain the *Shakura-Sunyaev viscosity* parameter at a disk radius r

$$\alpha = -\frac{T_{xy}}{q\Omega^2 \left(\frac{h}{r}\right)^3 M r^2}, \text{ where } M = \frac{\Omega x}{c_s}. \quad (9)$$

For a disk with $\zeta = 0.35$, $k_x = 3$, $k_y = 0.7$, a vertical secondary perturbation evolving for time $t_m = 10$ at a disk radius $r = 15$ and thickness $h(r)/r = 0.05$, $\alpha \sim 0.02$ (the detailed discussion is reported elsewhere⁸). This is interesting as α due to MRI computed by previous authors⁹ is similar order of magnitude.

4. Discussion

Above results verify that the three-dimensional growth rate due to the secondary perturbation in a Keplerian disk could be real and positive and corresponding growth may be exponential and significant enough to trigger elliptical instability. This eventually may trigger non-linearity and then plausible turbulence in the flow time scale. As this growth is the result of a three-dimensional perturbation, underlying perturbation effect should survive even in presence of viscosity. We also see that the corresponding viscosity α to transport matter inward and angular momentum outward is significant and comparable to that due to MRI.

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